Indian Statistical Institute, Bangalore Centre
M.Math II Year, First Semester
Mid-Sem Examination
Fourier Analysis

Time: 3 Hours September 10, 2012 Instructor: Pl.Muthuramalingam
Maximum mark you can get is: 40

1. Show that

   (a) $\sup_{\substack{f\in C_p[-\pi,\pi] \\ \|f\|_\infty\le 1}} \int_{-\pi}^{\pi} f(y)\ D_n(y) = \int_{-\pi}^{\pi} |D_n(y)|.$

   (b) Show that $\lim_{n\to\infty} \int_{-\pi}^{\pi} |D_n(y)| = \infty.$

   (c) Deduce from (b) that for a dense G_δ set in $C_p[-\pi,\pi]$ $\sup_n |(S_n f)(0)| = \infty.$ [6]

2. Let $f : [-\pi,\pi] \to R$ be continuous with $f(-\pi) = F(\pi)$ extend f to R by periodicity conditions. Assume that for some $\alpha > 1/2$

$$\int_{-\pi}^{\pi} \left| \frac{f(x+h)-f(x)}{h^\alpha} \right|^2 dx < \infty.$$

   Show that $\sum_{-\infty}^{\infty} |\hat{f}(n)| < \infty.$ [4]

3. Let

$$V_1 = \left\{ f : R \to \mathbb{C}, f(t) = \sum_{R=-\infty}^{\infty} a_k\ \chi_{[k,k+1]}(t) \right\}$$

   with $\sum |a_k|^2 < \infty$

$$V_0 = \left\{ g : R \to \mathbb{C}, g(t) = \sum_{k=-\infty}^{\infty} b_k\ \chi_{(\frac{k}{2},\frac{k+1}{2})}(t) \right\}$$

   with $\sum |b_k|^2 < \infty$.

   Find a relation between V_1 and V_0 like $V_0 \subset V_1$ or $V_1 \subset V_0$. Let W_0 be the orthogonal difference. Show that W_0 = closed linear space $\{\varphi(t-k) : k \in Z\}$ for a suitable orthonormal family $\{\varphi(t-k) : k \in Z\}$. [3]

4. Let $f \in L^2(R)$ be such that $Qf \in L^2(R)$ and $Pf \in L^2(R)$ when $(Qf)(t) = t\ f(t), (Pf)(t) = -i\ f'(t)$. Let $\psi \in C^\infty(R)$ be such that $\psi(t) = 1$ for $|t| \leq 1$ and 0 for $|t| \geq 2$, $0 \leq \psi \leq 1$. Define $f_n(t) = \psi(\frac{t}{n})\ f(t)$. Show that

$$\lim_{n\to\infty} \|f_n - f\|_2 + \|Qf_n - Qf\|_2 + \|Pf_n - Pf\|_2 = 0.$$

[5]

5. Let $f \in L'(R)$ with $f(0) = 0$ for $|t| \geq k$ for some k. Define $g : \mathbb{C} \to \mathbb{C}$ by $g(z) = \int\limits_{-k}^{k} dt\ e^{-itz} f(t)$. Show that g is analytic and calculate $g'(z)$.

[3]

6. Let $A = [-2\pi, -\pi] \cup [\pi, 2\pi]$. $e_k(t) = \frac{e^{-itr}}{\sqrt{2\pi}}$ for k in $\mathbb{Z}$. Show that $\{e_k\}$ is ONB for $L^2(A)$. [4]

7. Let $f \in L'[1, \infty)$. Show that there exists a_k in $[k, k+1]$ such that $f(a_k) \to 0$ as $k \to \infty$. Note that $a_k \to \infty$. [3]

8. Let $f \in L'(R)$, f is absolutely continuous and $f' \in L'(R)$. Find a relation between $\hat{f}(s)$ and $\hat{f}'(s)$ and prove your claim. [2]

9. Let μ be a complex valued measure on R with $\lambda(R) < \infty$ where

$$\lambda = \sup \left\{ \sum_j |\mu(E_j)| : E_1, E_2, \ldots \text{ is a partition of } R \right\}.$$

   (a) Show that $A = \{x : \mu(x) \neq 0\}$ is countable subset of R. [2]

   (b) Show that [3]

$$\lim_{T\to\infty} \frac{1}{2T} \int\limits_{-T}^{T} |\hat{\mu}(t)|^2\ dt = \sum_A |\ \mu\{x\}|^2.$$

10. Find Fourier transform of [5]

   (a) $e^{-(\frac{x^2}{2})}$

   (b) $x^n\ \ e^{-(\frac{x^2}{2})}$ for $n = 1, 2, 3 \ldots$

   (c) $\chi_{[a,b]}$

   (d) $\frac{t}{1+t^2}$. Note that this function is not in L'.